The Market and Individual Pricing Kernels Under No Arbitrage Asset Pricing Models

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Abstract: This chapter discusses how to use the No Arbitrage Asset Pricing Model (NAAPM) to determine the pricing kernel for both the financial markets and an individual with a given degree of constant risk aversion over her terminal wealth. The existence of the market and individual pricing kernel allows us to value any financial contract whose payoff is dependent on the prices from the NAAPM. Consequently, an individual would raise her lifetime utility by buying the assets which she prices higher than the financial market.

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1 Introduction

The pricing of financial instruments is accomplished using a function which operates on the payoff of the instrument to determine the current market price. This chapter develops the pricing kernel given a No Arbitrage Asset Pricing Model (NAAPM) and applies the analysis to a standard term structure model. This pricing kernel is expressed as a Guassian function of the current pricing factors to represent the conditional expectation of the pricing kernel, and a log-normal probability distribution for the transitional probability from the current factors to the future random factors. This probability function is found by applying the Forward Kolmogorov Equation (FKE) which leads to a linear partial differential equation (PDE) whose solution is the transitional probability function. The solution to this PDE is a Guassian function such that the future factors have a log-normal distribution with mean zero and finite variance-covariance matrix. Thus the pricing kernel under a NAAPM has a conditional expected value, which is a Guassian function of the pricing factors, and a log-normally distributed transitional probability from the current pricing factors to the future value of these factors.

The valuation of a financial instrument by an individual is also developed given the NAAPM. Suppose an individual has a constant relative risk return over wealth with a given investment horizon and a leverage restriction. In addition, the holding period return follows the NAAPM. In this case, the portfolio analysis of Sangvinatsos and Wachter (2005) and Liu (2007) is extended to incorporate the investor's leverage restriction. This analysis yields a portfolio rule which is a linear relation in the expected holding period return under the NAAPM, the leverage restriction, and the elasticity of the lifetime utility with respect to the pricing factors. Given the portfolio rule, the expected lifetime utility of the investor is the solution of a linear PDE. This solution is a Guassian function of the current pricing factors, so that the portfolio rule is linear in only the pricing factors. With the solution for the investor's expected lifetime utility and portfolio rules, Ito's lemma is used to derive the stochastic process for the investor's wealth and the lifetime utility. These stochastic processes have the same functional form as the pricing kernel for the NAAPM. Consequently, the exact same procedure is applied to split these stochastic processes into a Guassian conditional expectation and a normal transitional probability from the current pricing factors to the future value of these factors. Finally, the intertemporal rate of substitution from the current pricing factors to the future factors. This corresponds to a pricing kernel for an individual which is independent of the investor's wealth. In addition, the pricing kernel is the product of a Guassian function of the current pricing factors and a log-normal transitional probability for the investor's lifetime utility. Consequently, any financial contract, which is a function of the assets priced under the NAAPM, can have both a market price and an individual price. If the individual values the asset more (less) than the market, then it would add (deduct) value to the individual's lifetime utility.

The NAAPM was developed by Duffie, Pan and Singleton (2000) with initial application to the term structure by Duffie and Kan (1996), and Duffie and Singleton (1997). Dai and Singleton

(2000, 2002) developed the identification strategy for estimating term structure models. A complete survey of this research can be found in Piazzesi (2010). Joslin, Singleton and Zhu (2011), and Hamilton and Wu (2012a, 2014a) introduce procedures to improve the estimation of these models. For example Joslin, Singleton and Zhu showed that the factors can be estimated using a VAR model, so that only yield curve and risk premium parameters need to be estimated by maximum likelihood estimation. Adrian, Crump, and Moench (2013) develop a three step regression procedure which focus on matching the holding period return rather than the yield to maturity. The pricing kernel developed in this chapter can be developed for any of these methods as long as the shocks to the yield curve factors are Guassian.[3]

The NAAPM has been used to study: 1.) The interconnection among the yield curve and macroeconomic variables;[4] 2.) Expected inflation and real rates on treasury securities;[5] 3.) Interpretation of monetary policy;[6] 4.) The zero lower bond and the yield curve;[7] 5.) Bond risk premium;[8] 6.) Crude oil future prices;[9] 7.) Swap rates and credit quality;[10] 8.) Derivatives for fixed income securities.[11] 9.) Quantitative easing and the term structure.[12] The pricing kernel for NAAPM developed here can be used to price all these financial instruments and help to interpret their properties.

2 The Market Pricing Kernel

Consider the typical No Arbitrage Asset Pricing Model (NAAPM), which postulates that several latent factors drive all returns on marketable securities, in such a coherent way that no arbitrage is permitted.[13] These underlying latent factors, $X(s)$, are typically assumed to follow a mean-reverting stochastic process. Specifically, the dynamics for the factors under the physical probability distribution are given by

$$dX(s) = \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}} X(s)\right) ds + \Sigma_X d\epsilon_s. \tag{1}$$

Where, the N by 1 vector $X(s)$ contains N latent factors, the standard Brownian motion ϵ_s summarizes the uncertainty in the interest rate factors $X(s)$. The vector $\gamma^{\mathcal{P}}$ and the

[3] If time varying variance-covariance matrices are introduced then all the PDEs in this chapter will be more complicated in that the coefficients on the second order derivatives will not be constant.

[4] See Ang and Piazzesi (2003), Joslin, Priebsch and Singleton (2014), and references in Bauer and Rudebusch (2017).

[5] Ang, Bekaer, and Wei (2008), and Chernov and Mueller (2012).

[6] Ang and Piazzesi (2033), and Ang, Boivin, Dong and Loo-Kung (2011).

[7] Hamilton and Wu (2012b, 2016), Krippner (2015), Bauer and Rudebusch (2016), and Wu and Xia (2016).

[8] Cochrane and Piazzesi (2005), Adrian, Crump, and Moench (2013), and Greenwood and Vayanos (2014).

[9] Hamilton and Wu (2014b).

[10] Duffie and Huang (1996), and Duffie and Singleton (1997).

[11] Grinblatt and Longstaff (2000), and Longstaff, Santa-Clara and Schwartz (2001).

[12] See Li and Wei (2013) for a survey.

[13] The NAAPM model is not limited to the yield on zero coupon bonds. Liu (2016), and Durham (2013) use these methods to model both bonds and stocks. Yung (2017), and Benson (2015) use the NAAPM approach to model foreign currency and its forward price.

matrix $A^{\mathcal{P}}$ contain model parameters under the physical probability measure. In particular, the stationary mean of the factors is given by $\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$, and $A^{\mathcal{P}}$ determines the speed of mean-reversion. The matrix $\Sigma_X\Sigma_X'$ is the variance-covariance matrix of the shocks, $d\epsilon_s$, to the interest rate factors.

Because the dynamics of the latent interest rate factors are written in a continuous-time process while we observe data at discrete-time intervals, it is helpful to solve (1) for the interest rate factors, over the time interval τ, relative to its stationary value, $\bar{X}$,

$$X(t+\tau) - \bar{X} = e^{-A^{\mathcal{P}}\tau}\left(X - \bar{X}\right) + Y_\tau, \tag{2}$$

where

$$Y_\tau = \int_0^\tau e^{-A^{\mathcal{P}}(\tau - s)}\Sigma_X d\epsilon_s. \tag{3}$$

The first term in (2) captures the part of the deviation of the current interest rate factors from its stationary value that is expected to mean-revert as long as all eigenvalues of the matrix $A^{\mathcal{P}}$ are positive. The second term is the random shock to the interest rate factors from time t to $t+\tau$. This random shock can be shown to have a normal probability distribution with mean 0 and variance covariance matrix $\sigma_Y(\tau)$.[14]

For the purpose of this chapter let the assets be zero coupon Treasury securities with yield to maturity $r_{\tau,s}\left(X(s)\right)$, where s is the time at which the yield is observed and τ is the maturity of the yield. The Treasury yield to maturity is specified as an affine function of the latent factors $X(s)$

$$r_{\tau,s}\left(X(s)\right) = A_\tau + B_\tau X(s). \tag{4}$$

Where, the matrices of parameters A_τ and B_τ for each yield to maturity are solutions to a set of differential equations in a coherent way so that no arbitrage opportunity is permitted for investors in the financial markets.

To understand the affine structure of NAAPM, note that all yields to maturity conceptually depend on the risk free rate and the risk premium. First, the risk free interest rate $r(s)$ is assumed to be a linear function of the latent factors:

$$r(s) \equiv r\left(X(s)\right) = \delta_0 + \delta_1 X(s). \tag{5}$$

Here, the scalar δ_0 and the vector δ_1 are model parameters.

Furthermore, the risk price in the NAAPM is also assumed to be affine in the latent factors.

$$\Lambda\left(X(s)\right) = \lambda_0 + \lambda_1 X(s), \tag{6}$$

This specification of the risk price yields the so-called essentially affine model leading to the affine structure in (4).

[14] See Arnold (1974) for proof. The variance-covariance matrix is the solution to a Ricatti differential equation. The solution is found by using recursive rules, which are implemented in the lyap subroutine in Matlab with inputs $A^{\mathcal{P}}$ and Σ_X.

The model specifications so far imply a risk-neutral probability distribution of the latent factors through a change of measure which accounts for the price of risk. As a result, the dynamics of the process for the factors, $X(s)$, under the risk-neutral distribution, is

$$dX(s) = \left(\gamma^{\mathcal{Q}} - A^{\mathcal{Q}} X(s)\right) ds + \Sigma_X d\epsilon_s^{\mathcal{Q}}. \tag{7}$$

Note first that the variance-covariance matrix in this risk-neutral process remains the same as in the physical process, $\Sigma_X \Sigma_X'$. However, the vector $\gamma^{\mathcal{Q}}$ and the matrix $A^{\mathcal{Q}}$ are the risk adjusted parameters of the corresponding parameters in the physical process, through a change of variable using the risk price

$$\gamma^{\mathcal{Q}} = \gamma^{\mathcal{P}} - \Sigma_X \lambda_0 \text{ and } A^{\mathcal{Q}} = A^{\mathcal{P}} + \Sigma_X \lambda_1. \tag{8}$$

The yields to maturity are affine functions of latent factors implies that the bond prices are exponentially affine in latent factors

$$P_{\tau,s} = \exp\left[a_\tau + b_\tau \cdot X(s)\right]. \tag{9}$$

Where, $a_\tau = -\tau A_\tau$ and $b_\tau = -\tau B_\tau$.
The holding period return on a zero coupon bond maturing at τ is given by

$$\frac{dP_{\tau,s}}{P_{\tau,s}} = \left[b_\tau \left((\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - (A^{\mathcal{P}} - A^{\mathcal{Q}})X(s)\right) + r(s)\right] ds + b_\tau \Sigma_X d\epsilon_s. \tag{10}$$

The pricing kernel under NAAPM is given by

$$\begin{aligned} \frac{M_{\tau,t}}{M_{t,t}} &= \exp\left\{ -\int_t^{t+\tau} \left[r\left(X(s)\right) + \frac{1}{2}\Lambda\left(X(s)\right)' \Lambda\left(X(s)\right) \right] ds + \int_t^{t+\tau} \Lambda\left(X(s)\right)' d\epsilon_s \right\} \\ &= \exp\left\{ \int_0^{\tau} \left(-\mathcal{M}_1 - \frac{1}{2}\left(X_s' \mathcal{M}_3 X_s - 2\mathcal{M}_2 X_s \right) \right\} \right] ds + \int_t^{T} \left(\mathcal{M}_4 + \mathcal{M}_5 X_s\right) d\epsilon_s \right\}. \end{aligned} \tag{11}$$

We use the risk free rate, the risk premium and the risk neutral coefficients in this derivation, so that the constants are given by

$$\begin{aligned} \mathcal{M}_1 &\equiv \delta_0 + \frac{1}{2}\left(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}\right)' \left(\Sigma_X' \Sigma_X\right)^{-1} \left(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}\right), \\ \mathcal{M}_2 &\equiv -\left[\delta_1 - \left(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}\right)' \left(\Sigma_X' \Sigma_X\right)^{-1} \left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right)\right], \\ \mathcal{M}_3 &\equiv \left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right)' \left(\Sigma_X' \Sigma_X\right)^{-1} \left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right), \\ \mathcal{M}_4 &\equiv \left(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}\right)' \left(\Sigma_X'\right)^{-1} \text{ and } \mathcal{M}_5 \equiv -\left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right)' \left(\Sigma_X'\right)^{-1}. \end{aligned}$$

The pricing kernel is a random variable dependent on the solution to the yield curve factors (2). We want to find the probability density function for the pricing kernel (11) using the

Forward Kolmogorov Equation (FKE). Represent the transition probability from state X at time t to the state Y at time T by $p(t,X,T,Y)$. For the stochastic process (11) let

$$\phi(t,s)=\exp\left\{-\frac{1}{2}\int_t^s\left[X_v'\mathcal{M}_3(v)X_v-2\mathcal{M}_2(v)X_v\right]dv\right\}.$$

For fixed (t,X) the function

$$g(\tau,Y)\equiv\phi(t,\tau)p(t,X,\tau,Y) \tag{12}$$

solves the FKE.[15]

$$\frac{\partial g(\tau,Y)}{\partial\tau}=\mathcal{K}_Y^*g(\tau,Y)-\frac{1}{2}\left(Y'\mathcal{M}_3(\tau)Y-2\mathcal{M}_2(\tau)Y\right)g(\tau,Y). \tag{13}$$

Here, the dual of $\mathcal{K}_X$ given by[16]

$$\begin{aligned}\mathcal{K}_X^*&=-\sum_{i=1}^N\frac{\partial}{\partial X_i}\left(\gamma^{\mathcal{P}}-A^{\mathcal{P}}X\right)_i+\frac{1}{2}\sum_{i,j=1}^N\frac{\partial^2}{\partial X_i\partial X_j}\Sigma_{ik}\Sigma_{kj}'\\&=-\gamma^{\mathcal{P}\prime}\frac{\partial}{\partial X}+X'A^{\mathcal{P}\prime}\frac{\partial}{\partial X}+Trace(A^{\mathcal{P}})+\frac{1}{2}Trace\left(\Sigma\Sigma'\frac{\partial^2}{\partial X\partial X}\right).\end{aligned} \tag{14}$$

Remark: Notice that only the distribution of the factors enters (14). The preferences of the investor only influences the discount factor $\phi(t,\tau)$.

To find the initial condition, let the Dirac distribution centered at $X\in\mathbb{R}^N$ be $f(X)=\delta_X$ such that

$$\delta_X(\theta)=\int_{\mathbb{R}^N}\delta_x(Y)\theta(Y)dY=\theta(X).$$

For a given $X_t=X\in\mathbb{R}^N$,

$$g(\tau,X)=\int_{\mathbb{R}^N}\delta_X(Y)\phi(t,\tau)p(t,X,\tau,Y)dY=\phi(t,\tau)p(t,X,\tau,X).$$

Consequently, if the initial condition for the Kolmogorov forward equation (13) is

$$\lim_{\tau\to0^+}g(\tau,X(\tau))=\delta_X, \tag{15}$$

then the solution to (13) is $\phi(t,\tau)p(t,X,\tau,Y)=g(\tau,Y)$.

Thus, we have

[15] See Karatzas and Shreve (1988, p. 369) equation (7.24). Also see Theorem 8.7.1. of Calin, Chang, Furutani, and Iwasaki (2011), and Chirikjian (2009, p.118-121)

[16] See Øksendal (2005, p. 169). Also follow the derivation in Chirikjian (2009, p. 121)

Theorem 2.1. *The discounted transition probability $\phi(t,\tau)p(t,X,\tau,Y)$ for a given $X_t = X \in \mathbb{R}^N$ is the solution to the Kolmogorov Forward equation (13) with (14) subject to the initial condition (15).*

Proof. See Appendix. ∎

The solution to the FKE is difficult to find given the Dirac initial condition (15). To circumvent this problem we use the Fourier transform of the FKE problem, since the Fourier transform of (15) is 1.

Suppose that $f(X) \in \mathcal{S}(\mathbb{R}^N)$, on $\mathbb{R}^N$. This functional space refers to all functions which rapidly decrease, so that $f(X)$ is absolutely integrable over $\mathbb{R}^N$. This allows one to move between Fourier transforms and its inverse. The Fourier transform of $f(X)$ is

$$F[f(X)] = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(X)e^{-i\xi' X}\, dX. \tag{16}$$

Here $\xi \in \mathbb{R}^N$ and $\xi \cdot X \equiv \xi' X = \xi_1 X_1 + \cdots + \xi_N X_N$.

The inverse Fourier transform of $\hat{f}(\xi)$ is

$$F^{-1}[\hat{f}(\xi)] = f(X) = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi\cdot X}\, d\xi. \tag{17}$$

In the appendix we apply the Fourier transform to the FKE problem to yield the linear partial differential equation

$$\begin{aligned}&\frac{\partial F\left[g(\tau,Y)\right]}{\partial \tau} + \frac{1}{2}\xi'\Sigma\Sigma'\xi F[g(\tau,Y)] + i\gamma^{\mathcal{P}\prime}\xi F\left[g(\tau,Y)\right] \\ &- \left(\frac{\partial F[g(\tau,Y)]}{\partial \xi}\right)'\left(i\mathcal{M}_2(\tau)' - A^{\mathcal{P}\prime}\xi\right) + \frac{1}{2}Trace\left(\mathcal{M}_3(\tau)\frac{\partial^2 \hat{g}(\xi)}{\partial\xi\partial\xi}\right) = 0\end{aligned} \tag{18}$$

subject to the initial condition

$$F\left[g(0,Y_0)\right] = 1.$$

We use a guess and verify procedure to find its solution.

$$F\left[g(\tau,Y)\right] = \exp\left\{-\frac{1}{2}\left[\xi'\mathcal{G}_3(\tau)\xi - 2i\mathcal{G}_2(\tau)'\xi + \mathcal{G}_1(\tau)\right]\right\}, \tag{19}$$

We do not have to assume the matrix is symmetric, since $\frac{1}{2}\xi'\left(\mathcal{G}_3(\tau) + \mathcal{G}_3(\tau)'\right)\xi = \xi'\mathcal{G}_3(\tau)\xi$.

The coefficients in (19) are the solution to three ordinary differential equations (ODE).

$$\frac{\partial \mathcal{G}_3(\tau)}{\partial \tau} = \mathcal{G}_3(\tau)\mathcal{M}_3(\tau)\mathcal{G}_3(\tau) - 2\mathcal{G}_3(\tau)A^{\mathcal{P}\prime} + \Sigma_X\Sigma_X' \tag{20}$$

subject to

$$\mathcal{G}_3(0) = 0_{N\times N}.$$

$$\frac{\partial \mathcal{G}_2(\tau)}{\partial \tau} = \mathcal{M}_2(\tau)\left(\mathcal{M}_3(\tau)\mathcal{G}_3(\tau) - A^{\mathcal{P}\prime}\right) - \gamma^{\mathcal{P}\prime} - \mathcal{M}_2(\tau)\mathcal{G}_3(\tau) \tag{21}$$

subject to

$$\mathcal{G}_2(0) = 0_N.$$

$$\frac{\partial \mathcal{G}_1(\tau)}{\partial \tau} = 2\mathcal{G}_2(\tau)\mathcal{M}_2(\tau)' - \mathcal{G}_2(\tau)\mathcal{M}_3(\tau)\mathcal{G}_2(\tau)' - Trace\left(\mathcal{M}_3(\tau)\mathcal{G}_3(\tau)\right) \tag{22}$$

subject to

$$\mathcal{G}_1(0) = 0.$$

The solutions to these three ODEs are found using the ODE solver in Matlab. Given the solution to the Fourier transform to the FKE problem, the inverse Fourier transform yields the solution to the FKE problem.

$$g(\tau, Y) = \frac{1}{\sqrt{(2\pi)^N det(\mathcal{G}_3(\tau))}} \exp\left\{ -\frac{1}{2}\mathcal{G}_1(\tau) - \frac{1}{2}\left(\mathcal{G}_2(\tau)' - Y\right)' \mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)' - Y\right)\right\}. \tag{23}$$

Thus, the discounted transition probability is a Guassian function of the future yield curve factors Y.

The final step in determining both the conditional expectation and probability distribution for the pricing kernel is to use the solution to the random factors (2) in (11) to express the pricing kernel in terms of the current factors X and the random future factors Y_s for $t < s < T$. In the Appendix it is shown that the pricing kernel is given by

$$\frac{M_{\tau,t}}{M_{t,t}} = \mathcal{M}(\tau, X)\exp\left\{ -\frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s + \int_t^{t+\tau}\left(\mathfrak{M}_4 + X_t'\mathfrak{M}_5 + Y_s'\mathcal{M}_5\Sigma_X'\right)d\epsilon_s\right\}. \tag{24}$$

Here, the conditional pricing kernel is derived in the Appendix, and is given by

$$\mathcal{M}(\tau, X) \equiv \exp\left\{ -\frac{1}{2}\left(X - \mu_{\mathcal{M}}(\tau)\right)' \sigma_{\mathcal{M}}^{-1}\left(X - \mu_{\mathcal{M}}(\tau)\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1\right\} \tag{25}$$

$$
\begin{aligned}
\mu_{\mathcal{M}}(\tau) \equiv & \mathfrak{M}_3^{-1}\mathfrak{M}_2, \qquad \sigma_{\mathcal{M}} \equiv \mathfrak{M}_3^{-1} \\
\mathfrak{M}_1 \equiv & -\mathcal{M}_1(\tau)\tau - \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau \\
& + \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I - e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
& - \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}, \\
\mathfrak{M}_2 \equiv & \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right] + \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right] \\
& - \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right], \\
\mathfrak{M}_3 \equiv & \mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}, \\
\mathfrak{M}_4 \equiv & \Sigma_X\mathcal{M}_5'\left[I - e^{-A^{\mathcal{P}}(\tau-t)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \text{ and } \mathfrak{M}_5 \equiv e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_5.
\end{aligned}
$$

In this case the probability distribution of Y_τ is normally distributed with mean 0 and variance covariance matrix $\sigma_Y(\tau)$, where σ_Y is the solution to the Lypunov equation By exercise (1.2.11) of Hijab (1987)

$$
\sigma_Y(\tau) = \sigma_{Y\infty} - e^{-A^{\mathcal{P}}\tau}\sigma_{Y\infty}e^{-A^{\mathcal{P}\prime}\tau}.
$$

Here, the matrix $\sigma_{Y\infty}$ solves the Lyapunov equation

$$
-A^{\mathcal{P}}\sigma_{Y\infty} - \sigma_{Y\infty}A^{\mathcal{P}\prime} = \Sigma_X\Sigma_X'.
$$

As the time horizon tends to infinity, $\sigma_Y(\tau) \to \sigma_{Y\infty}$ given the eigenvalues of $A^{\mathcal{P}}$ are positive. The solution to this equation is a positive definite symmetric matrix, which is easily calculated using lyap.m in Matlab.

For solving the FKE we use $A^{\mathcal{P}} = 0$ and $\gamma^{\mathcal{P}} = 0$, since Y_τ is white noise. In addition, $\mathcal{M}_2 = 0$, since a linear term in Y_τ is not present in (24). In this case, $\mathcal{G}_2(\tau) = 0$ is the solution to (21). It is also the case that $-\mathcal{G}_1(\tau) > 0$ by (22) in this situation, since $\mathfrak{M}_3 > 0$. As a result, the transition probability in (23) has a log-normal distribution with with parameters 0 and σ_M given by the solution to (20). Thus, the conditional probability distribution for the pricing kernel is

$$
\frac{M_{\tau,t}}{M_{t,t}} = \frac{\mathcal{M}(\tau, X)}{\sqrt{(2\pi)^N \, det(\sigma_{\mathcal{M}})}} \exp\left\{ -\frac{1}{2}Y'\sigma_M^{-1}Y \right\}. \tag{26}
$$

Here, we include the expected value of this distribution, $e^{-\frac{1}{2}\mathcal{G}_1(\tau)}$, in $\mathfrak{M}_1 > 0$, since they are both constants independent of X and Y.

Thus, the financial market represented by the NAAPM yields a pricing kernel (25) and (26). This pricing kernel is a combination of a Guassian form in the current factors for the conditional expectation and a log-normal probability function for the future factors.

3 Investor's Pricing Kernel

To find the pricing kernel for an investor given holding period returns follows a NAAPM we examine the optimal behavior of an investor. The investor is assumed to be risk averse with a constant relative risk aversion utility (CRRA) with parameter γ. This investor maximizes the expected utility from terminal capital at a fixed time $\tau = T - t$ given her current wealth, $W(t) = W$ and yield curve factors, $X(t) = X$. The investment horizon of this investor is τ. The investor's conditional expected value is

$$J(W, X, \tau, t) = e^{-\beta\tau} E\left[\frac{(W(\tau))^{1-\gamma}}{1-\gamma}\middle| W(t) = W, X(t) = X\right], \tag{27}$$

where β is the discount rate for the investor.

Suppose the investor trades N marketable securities such that

$$\omega(s)'\iota + \omega_{1\tau}(s) = \xi \tag{28}$$

for $s \in [t, t+\tau]$. We use the following vector notation

$$\omega'(s) = \begin{pmatrix} \omega_{2\tau}(s) \cdots \omega_{N\tau}(s) \end{pmatrix}, \quad b' = \begin{pmatrix} b_{2\tau} \cdots b_{N\tau} \end{pmatrix} \text{ and } \iota' = \begin{pmatrix} 1 \cdots 1 \end{pmatrix}. \tag{29}$$

Here, ξ is the leverage ratio, so that $1 - \xi$ represents the amount of wealth $W(s)$ invested in the risk free asset, $\omega_i(s)$ is the percentage wealth invested in assets $i = 1, \cdots N$.

The change in the investor's wealth is

$$\frac{dW(s)}{W(s)} = (1-\xi)\mu_{1\tau}(s) + \sum_{i=2}^{N}\mu_{i\tau}(s)\omega_i(s) + \sum_{i=1}^{N}\omega_i(s)b'_{i\tau}\Sigma_X d\epsilon_s. \tag{30}$$

The instantaneous expected excess rates of return on marketable securities, from (10), are

$$\begin{aligned} \mu_{1\tau}(s) - r(s) \equiv & b'_\tau\left[(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - (A^{\mathcal{P}} - A^{\mathcal{Q}})X(s)\right] \\ \mu_{i\tau}(s) - \mu_{1\tau}(s) \equiv & (b'_{i\tau} - b'_\tau)\left[(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - (A^{\mathcal{P}} - A^{\mathcal{Q}})X(s)\right], i = 2, \cdots, N. \end{aligned} \tag{31}$$

The investor's problem extends the analysis of Sangvinatsos and Wachter (2005) and Liu (2007) to account for the leverage constraint. Chami, Cosimano, Jun and Rochon (2017) uses their procedure to derive the solution to the individual's problem.

$$\begin{aligned} J(W, X, \tau, t) = & \frac{(W(t))^{1-\gamma}}{1-\gamma}h(\tau, X)^\gamma, \\ & \text{where } h(\tau, X) = h(\tau)\exp\left\{-\frac{1}{2}(X - \mu_J(\tau))'(\sigma_J(\tau))^{-1}(X - \mu_J(\tau))\right\}. \end{aligned} \tag{32}$$

Given the solution, the individual's portfolio rule is given by

$$\begin{aligned}
\omega(t) =&\omega_1 \left\{(b-\iota b_\tau)\left[(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}})\ -(A^{\mathcal{P}}-A^{\mathcal{Q}})X(t)\right]\right\}+\omega_2\xi+\omega_3\gamma\left(\sigma_J(\tau)\right)^{-1}\left[X-\mu_J(\tau)\right] \\
\omega_1 \equiv&\left[\gamma\left(b\Sigma_X\Sigma_X'b'+\iota\iota'b_\tau\Sigma_X\Sigma_X'b_\tau'-2b\Sigma_X\Sigma_X'b_\tau'\iota'\right)\right]^{-1} \text{ with } \iota'=\begin{pmatrix} 1, & \cdots, & 1 \end{pmatrix}, \\
\omega_2 \equiv&2\omega_1\left(b\Sigma_X\Sigma_X'b_\tau'-\iota b_\tau\Sigma_X\Sigma_X'b_\tau'\right) \text{ and } \omega_3\equiv\omega_1\left(b-\iota b_\tau\right)\Sigma_X\Sigma_X'. \\
&\omega_1(t)=\xi-\iota'\omega(t).
\end{aligned} \tag{33}$$

Consequently, the portfolio rule is linear in the yield curve factors.

Given the lifetime utility of the investor, the valuation of an investment by an individual can be analyzed. We can find the stochastic process for future lifetime utility by applying Ito's lemma to (32) given the stochastic process from wealth (30), the return on zero coupon bonds (31), the optimal portfolio rule (33) and the stochastic process for the yield curve factors (1).

$$\begin{aligned}
J(W,X(t+\tau),\tau)=&\frac{(W)^{1-\gamma}}{1-\gamma}h(0,X)^\gamma\exp\left\{\int_0^\tau\left(\mathcal{J}_1-\frac{1}{2}\Big(X(s)'\mathcal{J}_3X(s)-2\mathcal{J}_2\Big)\right)ds\right. \\
&\left.-\int_0^\tau\Big(\mathcal{J}_4+X(s)'\mathcal{J}_5\Big)d\epsilon_s\right\}.
\end{aligned} \tag{34}$$

The coefficients in this stochastic process are stated in Chami, Cosimano, Jun and Rochon (2017). The stochastic process for lifetime utility (34) also has the same functional form as (24) with different coefficients. Consequently, it can be split into a conditional expectation of lifetime utility as in (24).

$$\begin{aligned}
E_t\left(J(W,X(t+\tau),\tau)\right)\equiv&\mathcal{J}(W,X,\tau)=\frac{(W)^{1-\gamma}}{1-\gamma}h(0,X)^\gamma \\
&\times\mathcal{J}(\tau)\exp\left\{-\frac{1}{2}\Big(X-\mu_{\mathcal{J}}(\tau)\Big)'\left(\sigma_{\mathcal{J}}(\tau)\right)^{-1}\Big(X-\mu_{\mathcal{J}}(\tau)\Big)\right\},
\end{aligned} \tag{35}$$

and a transitional probability for lifetime utility from the current yield curve factors X at time t to the random yield curve factors Y at time $t+\tau$,

$$p_J\left(t,X,\tau,Y\right)=\frac{\exp\left\{-\frac{1}{2}Y'\sigma_J(\tau)^{-1}Y\right\}}{\sqrt{(2\pi)^N det\left(\sigma_J(\tau)\right)}}. \tag{36}$$

We can therefore write the future lifetime utility as

$$J(W,X(t+\tau),\tau)=\mathcal{J}(W,X,\tau)p_J\left(t,X,\tau,Y\right), \tag{37}$$

so that the future marginal utility of wealth is

$$\frac{\partial J(W,X(t+\tau),\tau)}{\partial W}=\frac{\partial\mathcal{J}(W,X,\tau)}{\partial W}p_J\left(t,X,\tau,Y\right). \tag{38}$$

The current marginal utility of wealth is given by

$$\frac{\partial J(W,X,\tau)}{\partial W} = \frac{(W)^{-\gamma}}{1-\gamma} h(0,X)^{\gamma}, \tag{39}$$

Thus, the intertemporal rate of substitution or pricing kernel for the individual investor is

$$\mathcal{P}(t,X,\tau,Y) = \frac{h(\tau,X)^{\gamma}}{h(0,X)^{\gamma}} \mathcal{J}(\tau) \exp\left\{-\frac{1}{2}\Big(X - \mu_{\mathcal{J}}(\tau)\Big)' (\sigma_{\mathcal{J}}(\tau))^{-1} \Big(X - \mu_{\mathcal{J}}(\tau)\Big)\right\} p_J\left(t,X,\tau,Y\right). \tag{40}$$

This corresponds to the pricing kernel for an investor with a given degree of risk aversion, γ and leverage ratio ξ, so that any financial payoff can be priced given the characteristics of the investor.

4 Conclusion

This chapter shows how the financial market would price financial assets under the No Arbitrage Asset Pricing Model (NAAPM). The analysis in this chapter can be used to address multiple financial economic problems. An example of this is Chami, Cosimano, Jun and Rochon (2017) which develops a model of a bank holding company (BHC) with an active trading desk. The trading desk invests in marketable securities, so as to maximize the expected lifetime utility subject to a leverage constraint which is imposed by the Chief Operating Officer (COO) of the BHC. The trading desk's wealth is given to her by the COO. Given the trading desk's closed form solution, the COO can solve the optimal decisions of the loan officer.

Chami, Cosimano, Rochon and Yung (2018) develop a model of the treasury market based on the NAAPM pricing kernel discussed here. In this work they analyze the relation between monetary policy and the term structure. It is shown that the impact of monetary policy is dependent on the current yield curve factors. In particular, if these factors are below the mean of the pricing kernel, then an increase in these factors lead to an increase in the pricing kernel rather than the traditional decrease. This property will lead to similar impacts of factors on the pricing kernel for the NAAPM so that problems, such as optimal corporate investment, and the evaluation of funding value adjustments by derivative dealers for swap books, can be influenced by the NAAPM factors.[17] Finally, Yung (2017), and Cosimano and Yung (2018) show how the NAAPM pricing kernel can be used to model and explain exchange rate movements.

[17]See Tevin and Whelan (2003), and Kothari, Lewellen and Warner (2017) for the first type of study and Anderson, Duffie and Song (2018) for the second problem.

Appendix

In this section we find the probability distribution for terms like (34). The yield curve factors follow the Ornstein-Uhlembeck process (3) in the paper.

$$dX(s) = \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}} X(s)\right) ds + \Sigma_X d\epsilon_s. \tag{41}$$

Following Arnold (1974) Theorem 8.2.2, the fundamental solution is

$$\Phi(s) = e^{-A^{\mathcal{P}}(s-t)}.$$

The solution to (41) is

$$X(\tau) = e^{-A^{\mathcal{P}}(\tau-t)} X(t) + \left(I - e^{-A^{\mathcal{P}}(\tau-t)}\right) \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} + \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-\upsilon)} \Sigma_X d\epsilon_\upsilon. \tag{42}$$

Here $\tau > t$.

Following Arnold (1974) Theorem 8.2.12 the integral

$$Y_\tau = \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-\upsilon)} \Sigma_X d\epsilon_\upsilon \sim N(Y; 0, K(\tau)). \tag{43}$$

Here, $N(Y; 0, K(\tau))$ represents a normal distribution with mean zero.

Its variance-covariance matrix is given by

$$K(\tau) = \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-\upsilon)} \Sigma_X \Sigma_X' e^{-A^{\mathcal{P}\prime}(\tau-\upsilon)} d\upsilon.$$

By exercise (1.2.11) of Hijab (1987)

$$K(\tau) = K_\infty - e^{-A^{\mathcal{P}}\tau} K_\infty e^{-A^{\mathcal{P}\prime}\tau}.$$

Here, the matrix K_∞ solves the Lyapunov equation

$$-A^{\mathcal{P}} K_\infty - K_\infty A^{\mathcal{P}\prime} = \Sigma_X \Sigma_X'.$$

As the time horizon tends to infinity, $K(\tau) \to K_\infty$. The solution to this equation is a positive definite symmetric matrix, which is easily calculated using lyap.m in Matlab.

We have encountered several stochastic processes that have the form

$$Z(X,\tau) = \exp\left\{ -\frac{1}{2} \int_t^T \left[X_s' \mathcal{D}_3(s) X_s - 2\mathcal{D}_2(s) X_s \right] ds + \int_t^T \left(\mathcal{D}_4(s) + X_s' \mathcal{D}_5(s)\right) d\epsilon_s \right\}. \tag{44}$$

In particular, see (34) in which $\mathcal{D}_i(s)$ are replaced by $\mathcal{J}_i(s)$ for $i = 1, 2, 3, 4, 5$. We use the notation X_s rather than $X(s)$, used in the text, to indicate that X is a stochastic process. In addition, the calculations are for a given terminal time T or time horizon τ.

We want $Z(X, \tau)$ to be a uniformly integrable martingale. We recognize that it is a stochastic exponential (Doléans-Dade exponential). See Protter (2005, pp. 84-89). In our case, we have a continuous stochastic process for the factor. As a result, we have

$$\mathcal{E}(X_t) = exp\left\{X_t - \frac{1}{2}\left[X, X\right]_t\right\},$$

where $[X, X]_t$ is the quadratic variation of $Z(X, \tau)$.

Theorem 45 of Protter (2005, p.141) demonstrates $Z(X, \tau)$ to be a uniformly integrable martingale as long as

$$E\left[exp\left\{\frac{1}{2}\left[X, X\right]_t\right\}\right] < \infty.$$

In this case, the quadratic variation includes all the terms associated with the variance-covariance matrix $\Sigma_X \Sigma_X'$. In this case the quadratic variation is

$$E\left\{\exp\left[\Big(\mathcal{D}_4(0) + X(s)'\mathcal{D}_5(0)\Big)'\Big(\mathcal{D}_4(0) + X(s)'\mathcal{D}_5(0)\Big)\right]\right\} < \infty. \tag{45}$$

This is called the Novikov's Criterion. Below we show these expectations are bounded for the investor's problem.

If this is true, then the stochastic process is given by

$$Z(X, \tau) = Z(X, 0)E_t\left[\exp\left\{\int_0^\tau \left(\mathcal{D}_1(0) - \frac{1}{2}\Big(X(s)'\mathcal{D}_3(0)X(s) - 2\mathcal{D}_2(0)\Big)\right)ds\right\}\right]. \tag{46}$$

For this stochastic process to have a solution, the Novikov condition (45) must be satisfied. In this case, the quadratic variation is dependent on the convergence of the stochastic process for X_s. Its solution is given by (42). The deterministic part of this solution is convergent, as long as $A^{\mathcal{P}}$ has all positive roots. The stochastic part Y includes all the terms associated with the variance-covariance matrix which is bounded by

$$K(\tau) = K_\infty - e^{-A^{\mathcal{P}}\tau}Ke^{-A^{\mathcal{P}\prime}\tau} \leq K \text{ with } \tau = T - t.$$

This together with the convergence of the solution X_s (42) assures the quadratic variation (45) exists.

We will now explain how the Backward and Forward Kolmogorov Equations apply to our problem. We then find the solution to these Kolmogorov equations.

The Backward Kolmogorov Equation

To solve for the expectation of the stochastic process (44) we use the backward Kolmogorov equation. We represent the transition probability from state X at time t to the state Y at time T by $p(t, X, T, Y)$. Subsequently, we will derive the transition probability using the forward Kolmogorov equation. In the text X is the vector of interest rate factors at the current time and Y is the random component of these factors at time T given by (43).

We now consider the conditional expectation of (44). As long as the Novikov's Criterion (45) holds, the conditional expectation of (44) is

$$\begin{aligned} f(t, X) = \int_{\mathbb{R}^N} \exp\Bigg\{ -\frac{1}{2}\int_t^T \Big[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s\Big]ds\Bigg\} \\ \times f(T, Y)p(t, X, T, Y)dY. \end{aligned} \tag{47}$$

We will show $f(t, X)$ for any $t \in [0, T]$ is the solution to the backward Kolmogorov equation

$$\begin{aligned} &\frac{\partial f(t, X)}{\partial t} - \frac{1}{2}\left(X'\mathcal{D}_3(t)X - 2\mathcal{D}_2(t)X\right) f(t, X) \\ &+ \left(\frac{\partial f(t, X)}{\partial X}\right)' \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right) + \frac{1}{2}Trace\left(\Sigma_X\Sigma_X'\frac{\partial^2 f(t, X)}{\partial X \partial X}\right) = 0 \end{aligned} \tag{48}$$

under the stochastic process (41).[18] We will be using in the subsequent argument the operator $\mathcal{K}_X$ defined by

$$\mathcal{K}_X \equiv \left(\frac{\partial}{\partial X}\right)' \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right) + \frac{1}{2}Trace\left(\Sigma_X\Sigma_X'\frac{\partial^2}{\partial X \partial X}\right) \tag{49}$$

so that

$$\frac{\partial f(t, X)}{\partial t} - \frac{1}{2}\left(X'\mathcal{D}_3(t)X - 2\mathcal{D}_2(t)X\right) f(t, X) + \mathcal{K}_X f(t, X). \tag{50}$$

The Kolmogorov backward PDE is solved subject to the terminal condition

$$\lim_{t \uparrow T} f(t, X) = f(X), \quad X \in \mathbb{R}^N. \tag{51}$$

Proof. Define the integrating factor

$$\phi(t, s) = \exp\Bigg\{ -\frac{1}{2}\int_t^s \Big[X_v'\mathcal{D}_3(v)X_v - 2\mathcal{D}_2(v)X_v\Big]dv\Bigg\}.$$

Let

$$Y_s = \phi(t, s)f(s, X_s) \quad s \in [t, T]$$

[18] This is a variation on the argument for Theorem 8.4.1 of Calin, Chang, Furutani and Iwasaki(2011). Also see Duffie (2001) Appendix E, and Karatzas and Shreve (1988, pp. 366-369).

which is a function of the solution to the stochastic differential equation for X (42). As a result, we can apply Theorem 6.3.1 of Shreve (2006). For a Borel measurable function $h(y)$ on $t \in [0, T]$, we have

$$E\left[h(X(T)) \mid \mathcal{F}(t)\right] = g(t, X(t)).$$

Under these conditions, Lemma 6.4.2 of Shreve (2006), the stochastic process $g(t, X(t))$ is a martingale. Now introduce the discount process

$$D(t) = \phi(0, t).$$

Define

$$Y(t, X) - E\left[\phi(t, T)h(X(T)) \mid \mathcal{F}(t)\right],$$

then

$$Y(t, X) = \phi(0, t)f(t, X)$$

is a martingale and satisfies the PDE (50). However, $f(t, X)$ is not a martingale.

To see the reason for the PDE (50), apply Ito's lemma to Y_s under the stochastic process (41) to yield

$$\begin{aligned} dY_s = &- \frac{1}{2}\left[X_s' \mathcal{D}_3(s) X_s - 2\mathcal{D}_2(s) X_s\right] \phi(t, s) f(s, X_s) ds + \phi(t, s) \frac{\partial f(s, X_s)}{\partial s} ds \\ &+ \phi(t, s) \left(\frac{\partial f(s, X_s)}{\partial X}\right)' \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}} X_S\right) ds + \frac{1}{2}\phi(t, s) Trace\left(\Sigma_X \Sigma_X' \frac{\partial^2 f(s, X_S)}{\partial X \partial X}\right) ds \\ &+ \phi(t, s) \left(\frac{\partial f(s, X_S)}{\partial X}\right)' \Sigma_X d\epsilon_s \end{aligned}$$

For Y_s to be a martingale the drift term must be zero. This property is satisfied by the PDE (50).

Since Y_s is a martingale we can integrate from t to T

$$\begin{aligned} \phi(t, T) f(T, X_T) - \phi(t, t) f(t, X_t) = &\int_t^T \phi(t, s) \left[\frac{\partial f(s, X_s)}{\partial s} - \frac{1}{2}\left(X_s' \mathcal{D}_3(s) X_s - 2\mathcal{D}_2(s) X_s\right)\right. \\ \times f(s, X_s) + \mathcal{K}_{X_s} f(s, X_s)\Big] ds + &\int_t^T \phi(t, s) \left(\frac{\partial f(s, X_s)}{\partial X_s}\right)' \Sigma_X d\epsilon_s \end{aligned}$$

We impose (48) subject to the terminal condition (51). In addition we can use the martingale property to take expectations, since Novikov's Criterion (45) is true.

$$f(t, X(t)) = E_t\Big[\phi(t, T) f(Y)\Big] + E_t\left[\phi(t, s) \left(\frac{\partial f(s, X_S)}{\partial X}\right)' \Sigma_X d\epsilon_s\right]$$

The second term is zero which leads to the result: Thus, solving the backward Kolmogorov equation (48) for $f(t, X)$ yields the expectation (47). ■

Solving the Backward Kolmogorov Equation

We set the terminal condition for the backward Kolmogorov equation

$$f(X) = \exp\left\{\frac{1}{2}X'\mathcal{D}_3X + \mathcal{D}_2X + \mathcal{D}_1\right\},$$

where $\mathcal{D}_i$ are constants for the terminal condition.

Guess the solution of (48) has the form

$$f(t, X) = \exp\left\{ -\frac{1}{2}\Big[X'\mathcal{F}_3(t)X - 2\mathcal{F}_2(t)X + \mathcal{F}_1(t)\Big]\right\}, \tag{52}$$

$$\frac{\partial f(t,X)}{\partial X} = f(t,X)\left[-\mathcal{F}_3(t)X + \mathcal{F}_2(\tau)'\right].$$

$$\frac{\partial^2 f(t,X)}{\partial X \partial X} = f(t,X)\Big(\mathcal{F}_3(t)XX'\mathcal{F}_3(t) - 2\mathcal{F}_3(\tau)X\mathcal{F}_2(t) + \mathcal{F}_2(t)'\mathcal{F}_2(\tau) - \mathcal{F}_3(\tau)\Big).$$

$$\frac{\partial f(t,X)}{\partial t} = f(\tau, X)\left[-\frac{1}{2}X'\frac{\partial \mathcal{F}_3(t)}{\partial t}X + \frac{\partial \mathcal{F}_2(t)}{\partial t}X - \frac{1}{2}\frac{\partial \mathcal{F}_1(t)}{\partial t}\right].$$

Now substitute these results into the Kolmogorov backward equation (48).

$$\begin{aligned}
&\left[-\frac{1}{2}X'\frac{\partial \mathcal{F}_3(t)}{\partial t}X + \frac{\partial \mathcal{F}_2(t)}{\partial t}X - \frac{1}{2}\frac{\partial \mathcal{F}_1(t)}{\partial t}\right] - \frac{1}{2}\left(X'\mathcal{D}_3(t)X - 2\mathcal{D}_2(t)X\right) \\
&+ \left[-X'\mathcal{F}_3(t) + \mathcal{F}_2(\tau)\right]\left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right) \\
&+ \frac{1}{2}Trace\left(\Sigma_X\Sigma_X'\Big(\mathcal{F}_3(t)XX'\mathcal{F}_3(t) - 2\mathcal{F}_3(\tau)X\mathcal{F}_2(t) + \mathcal{F}_2(t)'\mathcal{F}_2(\tau) - \mathcal{F}_3(\tau)\Big)\right) = 0
\end{aligned}$$

$$\begin{aligned}
&\left[-\frac{1}{2}X'\frac{\partial \mathcal{F}_3(t)}{\partial t}X + \frac{\partial \mathcal{F}_2(t)}{\partial t}X - \frac{1}{2}\frac{\partial \mathcal{F}_1(t)}{\partial t}\right] - \frac{1}{2}X'\mathcal{D}_3(t)X + \mathcal{D}_2(t)X \\
&- X'\mathcal{F}_3(t)\gamma^{\mathcal{P}} + X'\mathcal{F}_3(t)A^{\mathcal{P}}X + \mathcal{F}_2(\tau)\gamma^{\mathcal{P}} - \mathcal{F}_2(\tau)A^{\mathcal{P}}X + \frac{1}{2}X'\mathcal{F}_3(t)\Sigma_X\Sigma_X'\mathcal{F}_3(t)X \\
&- \mathcal{F}_2(t)\Sigma_X\Sigma_X'\mathcal{F}_3(t)X + \frac{1}{2}\mathcal{F}_2(t)\Sigma_X\Sigma_X'\mathcal{F}_2(t)' - \frac{1}{2}Trace\left(\Sigma_X\Sigma_X'\mathcal{F}_3(t)\right) = 0
\end{aligned}$$

Now equate quadratic, linear, and constant terms to obtain three ODEs.

$$\frac{\partial \mathcal{F}_3(t)}{\partial t} = \mathcal{F}_3(t)\Sigma_X\Sigma_X'\mathcal{F}_3(t) - \mathcal{D}_3(t) + 2\mathcal{F}_3(t)A^{\mathcal{P}} \tag{53}$$

subject to

$$\mathcal{F}_3(0) = \mathcal{D}_3.$$

This is the Lyapunov equation.

$$\frac{\partial \mathcal{F}_2(t)}{\partial t} = \mathcal{F}_2(t)\left(\Sigma_X\Sigma_X'\mathcal{F}_3(t) + A^{\mathcal{P}}\right) - \mathcal{D}_2(t) + \gamma^{\mathcal{P}\prime}\mathcal{F}_3(t) \tag{54}$$

subject to

$$\mathcal{F}_2(0) = \mathcal{D}_2.$$

This ODE is linear so that we can use integrating factor to solve for $\mathcal{F}_2(t)$. The Final ODE is

$$\frac{\partial \mathcal{F}_1(t)}{\partial t} = 2\mathcal{F}_2(\tau)\gamma^{\mathcal{P}} + \mathcal{F}_2(t)\Sigma_X\Sigma_X'\mathcal{F}_2(t)' - Trace\left(\Sigma_X\Sigma_X'\mathcal{F}_3(t)\right) \tag{55}$$

subject to

$$\mathcal{F}_1(0) = \mathcal{D}_1.$$

This initial value problem is the simplest since everything on the right hand side of the ODE is known.

The Forward Kolmogorov Equation

Following Karatzas and Shreve (1988) the solution to the backward Kolmogorov equation (48) $f(t, X)$ for fixed (T, Y) is

$$f(t, X) \equiv p(t, X, T, Y). \tag{56}$$

In addition, for fixed (t, X) the function

$$g(\tau, Y) \equiv \phi(t, \tau)p(t, X, \tau, Y) \tag{57}$$

solves the forward Kolmogorov equation.[19]

$$\frac{\partial g(\tau, Y)}{\partial \tau} = \mathcal{K}_Y^* g(\tau, Y) - \frac{1}{2}\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)g(\tau, Y). \tag{58}$$

Here, the dual of $\mathcal{K}_X$ given by[20]

$$\begin{aligned}\mathcal{K}_X^* &= -\sum_{i=1}^{N}\frac{\partial}{\partial X_i}\left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right)_i + \frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2}{\partial X_i \partial X_j}\Sigma_{ik}\Sigma_{kj}' \\ &= -\gamma^{\mathcal{P}\prime}\frac{\partial}{\partial X} + X'A^{\mathcal{P}\prime}\frac{\partial}{\partial X} + Trace(A^{\mathcal{P}}) + \frac{1}{2}Trace\left(\Sigma\Sigma'\frac{\partial^2}{\partial X \partial X}\right).\end{aligned} \tag{59}$$

To find the initial condition, let the Dirac distribution centered at $X \in \mathbb{R}^N$ be $f(X) = \delta_X$ such that

$$\delta_X(\theta) = \int_{\mathbb{R}^N}\delta_x(Y)\theta(Y)dY = \theta(X).$$

[19] See Karatzas and Shreve (1988, p. 369) equation (7.24). Also see Theorem 8.7.1. of Calin *et. al* (2011), and Chirikjian (2009, p.118-121)

[20] See Øksendal (2005, p. 169). Also follow the derivation in Chirikjian (2009, p. 121)

For a given $X_t = X \in \mathbb{R}^N$,

$$g(\tau, X) = \int_{\mathbb{R}^N} \delta_X(Y)\phi(t,\tau)p(t,X,\tau,Y)dY = \phi(t,\tau)p(t,X,\tau,X).$$

Consequently, if the initial condition for the Kolmogorov forward equation (13) is

$$\lim_{\tau \to 0^+} g(\tau, X(\tau)) = \delta_X, \tag{60}$$

then the solution to (13) is $\phi(t,\tau)p(t,X,\tau,Y) = g(\tau,Y)$.

Thus, we have the proof of Theorem 2.1

Proof. We will use the property of the dual for the Kolmogorov operator, $\mathcal{K}_Y$ given by

$$\int_{\mathbb{R}^N} \mathcal{K}_Y f(Y) g(Y) dY = \int_{\mathbb{R}^N} f(Y)\mathcal{K}_Y^* g(Y)dY. \tag{61}$$

We know from (47) that

$$\begin{aligned}
f(t,X) = &\int_{\mathbb{R}^N} \exp\left\{ -\frac{1}{2}\int_t^T \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s\right]ds \right\} \\
&\times f(Y)p(t,X,T,Y)dY \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \exp\left\{ -\frac{1}{2}\int_t^\tau \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s\right]ds \right\} \\
&\exp\left\{ -\frac{1}{2}\int_\tau^T \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s\right]ds \right\} \\
&f(Y)p(t,X,\tau,Z)p(\tau,Z,T,Y)dZdY \\
&= \int_{\mathbb{R}^N} \phi(t,\tau)f(\tau,Z)p(t,X,\tau,Z)dZ
\end{aligned}$$

The next to last step uses the Chapman-Kolmogorov equation for a Markov process[21] and the last step uses the definition of $f(t,X)$. As a result, we know for any $t < \tau \leq T$

$$f(t,X) = \int_{\mathbb{R}^N} f(\tau,Y)\phi(t,\tau)p(t,X,\tau,Y)dY. \tag{62}$$

[21]See Chirikjian (2009, p. 108) equation (4.16).

Next differentiate in τ

$$\begin{aligned}0 = \frac{\partial f(t,X)}{\partial \tau} &= \int_{\mathbb{R}^N} \frac{\partial f(\tau,Y)}{\partial \tau}\phi(t,\tau)p(t,X,\tau,Y)dY + \int_{\mathbb{R}^N} f(\tau,Y)\frac{\partial \phi(t,\tau)p(t,X,\tau,Y)}{\partial \tau}dY \\ &= \int_{\mathbb{R}^N} f(\tau,Y)\frac{\partial \phi(t,\tau)p(t,X,\tau,Y)}{\partial \tau}dY - \int_{\mathbb{R}^N} \mathcal{K}_Y f(\tau,Y)\phi(t,\tau)p(t,X,\tau,Y)dY \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} \left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right) f(\tau,X)\phi(t,\tau)p(t,X,\tau,Y)dY\end{aligned} \tag{63}$$

The second step uses the backward Kolmogorov equation (48).

Now apply the property (61) to find

$$\begin{aligned}0 = \int_{\mathbb{R}^N} f(\tau,Y)\bigg[&\frac{\partial \phi(t,\tau)p(t,X,\tau,Y)}{\partial \tau} - \mathcal{K}_Y^*(\phi(t,\tau)p(t,X,\tau,Y)) \\ &+ \frac{1}{2}\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)\phi(t,\tau)p(t,X,\tau,Y)\bigg]dY\end{aligned}$$

This means we want to define $g(\tau,Y) = \phi(t,\tau)p(t,X,\tau,Y)$ for (13).to hold.

■

Solving the Forward Kolmogorov Equation

It is difficult to impose the initial condition (15), since there is no explicit form for it. However, the Fourier transform of δ_X is 1. As a result, we will take the Fourier transform of the Kolmogorov equation (13) and find its solution. We will then apply the inverse Fourier transform to find the solution to the Kolmogorov forward equation given the initial condition.

If the Fourier transforms of $f(X)$ (16) exists, then

$$\begin{aligned}F_X\left[\frac{\partial f(X)}{\partial X_j}\right] &= i\xi_j F_X[f(X)] \Rightarrow F_X\left[\frac{\partial f(X)}{\partial X}\right] = i\xi F_X[f(X)]. \\ F_X\left[\frac{\partial^2 f(X)}{\partial X_j \partial X_k}\right] &= -\xi_j\xi_k F_X\left[f(X)\right] \Rightarrow F_X\left[\frac{\partial^2 f(X)}{\partial X \partial X}\right] = -\xi\xi' F_X[f(X)].\end{aligned} \tag{64}$$

The subscript X is added to keep track of the integration over X not t.

$$F_X[-iXf(X)] = \frac{\partial \hat{f}(\xi)}{\partial \xi} \Rightarrow F_X[Xf(X)] = i\frac{\partial \hat{f}(\xi)}{\partial \xi}. \tag{65}$$

Proof: $\frac{\partial \hat{f}(\xi)}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\int_{-\infty}^{\infty} f(X)e^{-i\xi\cdot X}\,dX = \int_{-\infty}^{\infty} -iX_j f(X)e^{-i\xi\cdot X}\,dX = F_X[-iX_j f(X)]$.

$\Rightarrow F_X[-iXf(X)] = \frac{\partial F_X\left[f(X)\right]}{\partial \xi}$.

$$
\begin{aligned}
F_X\left[\left(\frac{\partial f}{\partial X}\right)' A^{\mathcal{P}} X\right] &= Trace\left(A^{\mathcal{P}} F_X\left[X\left(\frac{\partial f}{\partial X}\right)'\right]\right) = iTrace\left(A^{\mathcal{P}}\frac{\partial F_X\left[\left(\frac{\partial f}{\partial X}\right)'\right]}{\partial \xi}\right)\\
&= i^2 Trace\left(A^{\mathcal{P}}\frac{\partial \xi' F_X[f(X)]}{\partial \xi}\right) = -Trace\left(A^{\mathcal{P}}\frac{\partial F_X[f(X)]}{\partial \xi}\xi' + A^{\mathcal{P}} F_X[f(X)]\right).
\end{aligned}
$$

The first result applies the Trace to a quadratic form. The second step uses (65) for the function $\left(\frac{\partial f}{\partial X}\right)'$. In the third equality we use the first result in (64). Finally, we use the product rule of differentiation and $i^2 = -1$.

We also have to consider $F_X[X'Xf(X)]$.

$$
F_X[X'Xf(X)] = \frac{\partial^2 \hat{f}(\xi)}{\partial \xi \partial \xi}
$$

Proof: $\frac{\partial \hat{f}(\xi)}{\partial \xi_j \partial \xi_k} = \frac{\partial}{\partial \xi_k}\int_{-\infty}^{\infty} -iX_j f(X)e^{-i\xi\cdot X}\,dX = \int_{-\infty}^{\infty} iX_k iX_j f(X)e^{-i\xi\cdot X}\,dX = F_X[-X_k X_j f(X)]$.

$\Rightarrow F_X[-XX'f(X)] = \frac{\partial^2 F_X\left[f(X)\right]}{\partial \xi \partial \xi}$.

Notice

$$
\begin{aligned}
F_X\left[X'\mathcal{D}_3(\tau)Xf(\tau,X)\right] &= F_X\left[Trace\left(X'\mathcal{D}_3(\tau)X\right)f(\tau,X)\right] = F_X\left[Trace\left(\mathcal{D}_3(\tau)XX'f(\tau,X)\right)\right]\\
&= Trace\left(F_X\left[\mathcal{D}_3(\tau)XX'f(\tau,X)\right]\right) = Trace\left(\mathcal{D}_3(\tau)\frac{\partial^2 \hat{f}(\xi)}{\partial \xi \partial \xi}\right)
\end{aligned}
$$

The first step is true since $X'\mathcal{D}_3(\tau)X \in \mathbb{R}$. The second step uses the property $Trace\,(ABC) = Trace\,(BCA)$. The third step takes advantage of the trace being a linear operator so that the additive property of integrals can be used. Since $X'X$ is symmetric the last step uses the last property of Fourier transforms.

Recall the Kolmogorov forward equation

$$
\begin{aligned}
\frac{\partial g(\tau,Y)}{\partial t} =& -\gamma^{\mathcal{P}\prime}\frac{\partial g(\tau,Y)}{\partial Y} + \left(\frac{\partial g(\tau,Y)}{\partial Y}\right)' A^{\mathcal{P}} Y + Trace\left(A^{\mathcal{P}}\right) g(\tau,Y)\\
&+ \frac{1}{2}Trace\left(\Sigma\Sigma'\frac{\partial^2 g(\tau,Y)}{\partial Y \partial Y}\right) - \frac{1}{2}\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)g(\tau,Y). \qquad (66)
\end{aligned}
$$

subject to the initial condition

$$
g(0,Y_0) = \delta_Y.
$$

Apply the Fourier transform to the forward Kolmogorov equation.

$$\begin{aligned}
\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} &= -\gamma^{\mathcal{P}\prime}F_Y\left[\frac{\partial g(\tau,Y)}{\partial Y}\right]+F_Y\left[\left(\frac{\partial g(\tau,Y)}{\partial Y}\right)' A^{\mathcal{P}}Y\right]\\
&+Trace(A^{\mathcal{P}})F_Y\left[g(\tau,Y)\right]+\frac{1}{2}Trace\left(\Sigma\Sigma' F_Y\left[\frac{\partial^2 g(\tau,Y)}{\partial Y\partial Y}\right]\right)\\
&-\frac{1}{2}F_Y\left[\left(Y'\mathcal{D}_3(\tau)Y-2\mathcal{D}_2(\tau)Y\right)g(\tau,Y)\right]
\end{aligned} \tag{67}$$

subject to the initial condition

$$F_Y\left[g(0,Y_0)\right]=1.$$

Next use the rules for Fourier transform to obtain

$$\begin{aligned}
\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} &= -i\gamma^{\mathcal{P}\prime}\xi F_Y\left[g(\tau,Y)\right]-Trace\left(A^{\mathcal{P}}\frac{\partial F_Y[g(\tau,Y)]}{\partial\xi}\xi'+A^{\mathcal{P}}F_Y[g(\tau,Y)]\right)\\
&+Trace(A^{\mathcal{P}})F_Y\left[g(\tau,Y)\right]-\frac{1}{2}Trace\left(\Sigma\Sigma'\xi\xi' F_Y[g(\tau,Y)]\right)-\frac{1}{2}Trace\left(\mathcal{D}_3(\tau)\frac{\partial^2\hat{g}(\xi)}{\partial\xi\partial\xi}\right)\\
&+i\left(\frac{\partial F_Y[g(\tau,Y)]}{\partial\xi}\right)'\mathcal{D}_2(t,\tau)'
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau}+\frac{1}{2}\xi'\Sigma\Sigma'\xi F_Y[g(\tau,Y)]+i\gamma^{\mathcal{P}\prime}\xi F_Y\left[g(\tau,Y)\right]\\
&-\left(\frac{\partial F_Y[g(\tau,Y)]}{\partial\xi}\right)'\left(i\mathcal{D}_2(\tau)'-A^{\mathcal{P}\prime}\xi\right)+\frac{1}{2}Trace\left(\mathcal{D}_3(\tau)\frac{\partial^2\hat{g}(\xi)}{\partial\xi\partial\xi}\right)=0
\end{aligned} \tag{68}$$

subject to the initial condition

$$F_Y\left[g(0,Y_0)\right]=1.$$

Now that the initial value problem is defined we can use a guess and verify procedure to find its solution.

$$F_Y\left[g(\tau,Y)\right]=\exp\left\{-\frac{1}{2}\left[\xi'\mathcal{G}_3(\tau)\xi-2i\mathcal{G}_2(\tau)'\xi+\mathcal{G}_1(\tau)\right]\right\}, \tag{69}$$

We do not assume the matrix is symmetric, since $\frac{1}{2}\xi'\left(\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right)\xi=\xi'\mathcal{G}_3(\tau)\xi$.

$$\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\xi}=F_Y\left[g(\tau,Y)\right]\left[-\mathcal{G}_3(\tau)\xi-\mathcal{G}_3(\tau)'\xi+i\mathcal{G}_2(\tau)\right].$$

$$\begin{aligned}
\frac{\partial^2 F_Y\left[g(\tau,Y)\right]}{\partial\xi\partial\xi} &= F_Y\left[g(\tau,Y)\right]\Big(-\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\xi'\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\\
&-2i\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\mathcal{G}_2(\tau)'-\mathcal{G}_2(\tau)\mathcal{G}_2(\tau)'-\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\Big).
\end{aligned}$$

$$\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} = F_Y\left[g(\tau,Y)\right]\left[-\frac{1}{2}\xi'\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}\xi + i\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}\xi - \frac{1}{2}\frac{\partial\mathcal{G}_1(\tau)}{\partial\tau}\right].$$

Now substitute these results into the Fourier transform (68) of the forward Kolmogorov equation (13).

$$\begin{aligned}
&\left[-\frac{1}{2}\xi'\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}\xi + i\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}\xi - \frac{1}{2}\frac{\partial\mathcal{G}_1(\tau)}{\partial\tau}\right] + \frac{1}{2}\xi'\Sigma_X\Sigma_X'\xi + i\xi'\gamma^{\mathcal{P}} \\
&- \left(-\xi'\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right] + i\mathcal{G}_2(\tau)\right)\left(i\mathcal{D}_2(\tau)' - A^{\mathcal{P}\prime}\xi\right) \\
&+ \frac{1}{2}Trace\Bigg(\mathcal{D}_3(\tau)\Big(\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\xi'\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right] \\
&- 2i\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\mathcal{G}_2(\tau) - \mathcal{G}_2(\tau)'\mathcal{G}_2(\tau) - \left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\Big)\Bigg) = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow &\left[-\frac{1}{2}\xi'\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}\xi + i\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}\xi - \frac{1}{2}\frac{\partial\mathcal{G}_1(\tau)}{\partial\tau}\right] + \frac{1}{2}\xi'\Sigma_X\Sigma_X'\xi + i\gamma^{\mathcal{P}\prime}\xi \\
&+ \mathcal{D}_2(\tau)\mathcal{G}_3(\tau)i\xi - \xi'\mathcal{G}_3(\tau)A^{\mathcal{P}\prime}\xi + \mathcal{G}_2(\tau)\mathcal{D}_2(\tau)' + \mathcal{G}_2(\tau)A^{\mathcal{P}\prime}i\xi + \frac{1}{2}\xi'\mathcal{G}_3(\tau)\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)\xi \\
&- \mathcal{G}_2(\tau)\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)i\xi - \frac{1}{2}\mathcal{G}_2(\tau)\mathcal{D}_3(\tau)\mathcal{G}_2(\tau)' - \frac{1}{2}Trace\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)\right) = 0.
\end{aligned}$$

Now equate quadratic, linear ($i\xi$), and constant terms to obtain three ODEs.

$$\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau} = \mathcal{G}_3(\tau)\mathcal{D}_3(\tau)\mathcal{G}_3(\tau) - 2\mathcal{G}_3(\tau)A^{\mathcal{P}\prime} + \Sigma_X\Sigma_X' \tag{70}$$

subject to

$$\mathcal{G}_3(0) = 0_{N\times N}.$$

Again this is the Lyapunov equation.

$$\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau} = \mathcal{G}_2(\tau)\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau) - A^{\mathcal{P}\prime}\right) - \gamma^{\mathcal{P}\prime} - \mathcal{D}_2(\tau)\mathcal{G}_3(\tau) \tag{71}$$

subject to

$$\mathcal{G}_2(0) = 0_N.$$

This ODE is linear so that we can use integrating factor to solve for $\mathcal{G}_2(\tau)$. The integrating factor is

$$int = e^{-\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau) - A^{\mathcal{P}\prime}\right)\tau}.$$

Consequently,

$$\frac{\partial e^{-\left(\mathcal{D}_3(s)\mathcal{G}_3(s)-A^{\mathcal{P}\prime}\right)s}\mathcal{G}_2(s)}{\partial s}ds = -e^{-\left(\mathcal{D}_3(s)\mathcal{G}_3(s)-A^{\mathcal{P}\prime}\right)s}\left(\gamma^{\mathcal{P}\prime} - \mathcal{D}_2(s,X)\mathcal{G}_3(s)\right)ds.$$

Now integrate from τ to 0

$$\mathcal{G}_2(\tau,X) = e^{\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)-A^{\mathcal{P}\prime}\right)\tau}\mathcal{G}_2(0) - \int_0^{\tau} e^{-\left(\mathcal{D}_3(s)\mathcal{G}_3(s)-A^{\mathcal{P}\prime}\right)(s-\tau)}\left(\gamma^{\mathcal{P}\prime} - \mathcal{D}_2(s,X)\mathcal{G}_3(s)\right)ds.$$

The Final ODE is

$$\frac{\partial \mathcal{G}_1(\tau)}{\partial \tau} = 2\mathcal{G}_2(\tau)\mathcal{D}_2(\tau)' - \mathcal{G}_2(\tau)\mathcal{D}_3(\tau)\mathcal{G}_2(\tau)' - Trace\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)\right) \tag{72}$$

subject to

$$\mathcal{G}_1(0) = 0.$$

This initial value problem is the simplest since everything on the right hand side of the ODE is known.

Solving these three ODEs leads to the solution (19) to the Fourier transform of the Kolmogorov equation (68). The final step is to take the inverse Fourier transform to (19)

$$g(\tau,Y) = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left[\xi'\mathcal{G}_3(\tau)\xi - 2\left(\mathcal{G}_2(\tau)-Y'\right)i\xi + \mathcal{G}_1(\tau)\right]\right\}d\xi. \tag{73}$$

To calculate this integral we use the following Lemma following Strauss (2008, p. 345) and Strichartz (2008, pp. 41-43).

Lemma 4.1. *Let α be a positive number and let x_0 and y_0 be real numbers.*

$$\int_{-\infty}^{\infty} e^{-\alpha(x+x_0+iy_0)^2}\,dx = \sqrt{\frac{\pi}{\alpha}} \tag{74}$$

We also need the multiple dimension version of Lemma 4.1.

Lemma 4.2. *Let A be a $N\times N$ symmetric matrix with all positive eigenvalues and let $Z\in\mathbb{C}^N$.*

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\left(X+A^{-1}Z\right)'A\left(X+A^{-1}Z\right)}dX = \sqrt{\frac{(2\pi)^N}{detA}}. \tag{75}$$

To apply the Lemma 4.2 to the inverse Fourier transform (73) we have to multiply out the quadratic exponent

$$\left(X+A^{-1}Z\right)'A\left(X+A^{-1}Z\right) = X'AX + 2Z'X + Z'(A^{-1})Z \tag{76}$$

Now match up the coefficients in (73) to yield

$$A = \mathcal{G}_3(\tau) \text{ and } Z = \left(\mathcal{G}_2(\tau)' - X\right)i \tag{77}$$

As a result, we can complete the square in the exponent of (73) to find

$$g(\tau,Y) = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\Big[\xi'\mathcal{G}_3(\tau)\xi - 2\left(\mathcal{G}_2(\tau)-Y'\right)i\xi + \mathcal{G}_1(\tau)\Big]\right\}d\xi$$
$$= \exp\left\{-\frac{1}{2}\mathcal{G}_1(\tau) - \frac{1}{2}\left(\mathcal{G}_2(\tau)'-Y\right)'\mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)'-Y\right)\right\}$$
$$\times\frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(Y+\mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)'-Y\right)i\right)'\mathcal{G}_3(\tau)\left(Y+\mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)'-Y\right)i\right)\right\}d\xi$$
$$= \frac{1}{\sqrt{(2\pi)^N\,det(\mathcal{G}_3(\tau))}}\exp\left\{-\frac{1}{2}\mathcal{G}_1(\tau) - \frac{1}{2}\left(\mathcal{G}_2(\tau)'-Y\right)'\mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)'-Y\right)\right\}. \tag{78}$$

By applying this solution to the forward Kolmogorov equation for the stochastic process (44), we can find the probability distribution for the investor's lifetime utility (34).

These random terms are probability densities of a normal distribution. We denote these probabilities densities by

$$\mathcal{N}(x;\mu,\Sigma) \equiv \frac{1}{\sqrt{(2\pi)^N\,det(\Sigma)}}\exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} \tag{79}$$

for $x \in \mathbf{R}^n$.

By (23) the discounted transition probability can be written as

$$\phi(t,\tau)p(t,X,\tau,Y) = \exp\left\{-\frac{1}{2}\mathcal{G}_1(\tau)\right\}\mathcal{N}\left(Y;\mathcal{G}_2(\tau)',\mathcal{G}_3(\tau)\right). \tag{80}$$

Note that

$$\phi(t,s) = \exp\left\{-\frac{1}{2}\int_t^s\Big[X_v'\mathcal{D}_3(v)X_v - 2\mathcal{D}_2(v)X_v\Big]dv\right\}$$

does not include the constant term

$$\mathcal{D}_0(\tau) = \exp\left\{-\frac{1}{2}\mathcal{D}_1(\tau)\tau\right\}$$

so it has to be added back in. The same is true for the backward Kolmogorov equation (48).

In the analysis of option values and VaR we will use various rules for Gaussian probability distributions which we recall from Petersen and Pedersen (2008). First we use the rule for the product of two normal distributions.

$$\begin{aligned}
\mathcal{N}\left(x;\mu_1,\Sigma_1\right)\times\mathcal{N}\left(x;\mu_1,\Sigma_1\right) &= \vartheta\mathcal{N}\left(x;\mu_c,\Sigma_c\right) \\
\text{where } \vartheta \equiv \frac{1}{\sqrt{\left(2\pi\right)^N det\left(\Sigma_1+\Sigma_2\right)}} &\exp\left\{-\frac{1}{2}(\mu_1-\mu_2)'\left(\Sigma_1+\Sigma_2\right)^{-1}\left(\mu_1-\mu_2\right)\right\}, \\
\mu_c &\equiv \left(\Sigma_1^{-1}+\Sigma_2^{-1}\right)^{-1}\left(\Sigma_1^{-1}\mu_1+\Sigma_2^{-1}\mu_2\right), \\
\text{and } \Sigma_c &= \left(\Sigma_1^{-1}+\Sigma_2^{-1}\right)^{-1}.
\end{aligned} \tag{81}$$

We also use the linear rule[22]

$$Ax \sim \mathcal{N}(x, A\mu, \Sigma A'), \tag{82}$$

Finally, we convert to a standard normal using the rule

$$x = \sigma Z + \mu \text{ such that } Z \sim \mathcal{N}\left(0_N, I_N\right). \tag{83}$$

Here, $\Sigma = \sigma\sigma'$ is the Cholesky decomposition of the variance covariance matrix. By following these basic rules for a normal distribution we are able to represent the probability distribution for the trading desk's bank capital and her lifetime utility.

Stochastic Discount Factor

We now have all the tools necessary to break a stochastic process like (44) into expected and random components. First, we apply the argument to the stochastic discount factor. The other stochastic processes will be solved using the same technique.

[22]See Petersen amd Pedersen (2008) 8.1.4, p. 41.

$$
\begin{aligned}
\frac{M_{\tau,t}}{M_{t,t}} =& \exp\left\{-\int_t^{t+\tau}\left[r\left(X(s)\right)+\frac{1}{2}\Lambda\left(X(s)\right)'\Lambda\left(X(s)\right)\right]ds+\int_t^{t+\tau}\Lambda\left(X(s)\right)'d\epsilon_s\right\} \\
=& \exp\Bigg\{-\int_t^{t+\tau}\Bigg[r\left(X(s)\right)+\frac{1}{2}\Bigg(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}-\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Bigg)'\left(\Sigma_X'\Sigma_X\right)^{-1} \\
&\Bigg(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}-\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Bigg)\Bigg]ds+\int_t^{t+\tau}\Bigg(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}-\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Bigg)'\left(\Sigma_X'\right)^{-1}d\epsilon_s\Bigg\} \\
=& \exp\Bigg\{-\int_t^{t+\tau}\Bigg[\delta_0+\frac{1}{2}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right) \\
&+\Bigg(\delta_1-\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)\Bigg)X(s) \\
&+\frac{1}{2}X(s)'\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Bigg]ds \\
&+\int_t^{t+\tau}\Bigg(\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1}-X(s)'\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1}\Bigg)d\epsilon_s\Bigg\} \\
=& \exp\left\{\int_0^{\tau}\Bigg(-\mathcal{M}_1(0)-\frac{1}{2}\Bigg(X_s'\mathcal{M}_3(0)X_s-2\mathcal{M}_2(0)X_s\Bigg)\right\}\Bigg]ds+\int_t^{T}\left(\mathcal{M}_4+\mathcal{M}_5X_s\right)d\epsilon_s\Bigg\}.
\end{aligned}
$$

We use the risk free rate, the risk premium and the risk neutral coefficients in this derivation.

The constants are given by

$$
\begin{aligned}
\mathcal{M}_1 \equiv& \delta_0+\frac{1}{2}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right), \\
\mathcal{M}_2 \equiv& -\left[\delta_1-\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)\right], \\
\mathcal{M}_3 \equiv& \left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right), \\
\mathcal{M}_4 \equiv& \left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1} \text{ and } \mathcal{M}_5 \equiv -\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1}.
\end{aligned} \tag{84}
$$

As a result, the stochastic process for the pricing kernel is

$$
\frac{M_{\tau,t}}{M_{t,t}} = \exp\left\{\int_0^{\tau}\Bigg(-\mathcal{M}_1(0)-\frac{1}{2}\Bigg(X_s'\mathcal{M}_3(0)X_s-2\mathcal{M}_2(0)X_s\Bigg)\Bigg)ds+\int_t^{t+\tau}\left(\mathcal{M}_4+X_s'\mathcal{M}_5\right)\Sigma_X'd\epsilon_s\right\}. \tag{85}
$$

We need the probability distribution for the pricing kernel in solving this stochastic process. Before applying the forward Kolmogorov results, we factor out all the deterministic terms from (85). We have from (42)

$$
X(\tau) = A_0(\tau)+e^{-A^{\mathcal{P}}(\tau-t)}X(t)+Y_\tau, \tag{86}
$$

where

$$A_0(\tau) = \left(I - e^{-A^{\mathcal{P}}(\tau - t)}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}.$$

We also will use

$$\int_t^{t+\tau} e^{-A^{\mathcal{P}}(s-t)} ds = \left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right].$$

Now factor the square term to find

$$\begin{aligned}
X(\tau)'\mathcal{M}_3 X(\tau) &= \left(A_0(\tau) + e^{-A^{\mathcal{P}}(\tau-t)}X(t) + Y_\tau\right)' \mathcal{M}_3 \left(A_0(\tau) + e^{-A^{\mathcal{P}}(\tau-t)}X(t) + Y_\tau\right) \\
&= \left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right) + X(t)' e^{-A^{\mathcal{P}\prime}}\right)\mathcal{M}_3 \\
&\left(\left(I - e^{-A^{\mathcal{P}}(\tau-t)}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} + e^{-A^{\mathcal{P}}(\tau-t)}X(t)\right) \\
&+ 2\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right) + X(t)' e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3 Y_\tau + Y_\tau' \mathcal{M}_3 Y_\tau \\
&= \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3\left(I - e^{-A^{\mathcal{P}}(\tau-t)}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&+ 2\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3 e^{-A^{\mathcal{P}}(\tau-t)}X(t) + X(t)' e^{-A^{\mathcal{P}\prime}}\mathcal{M}_3 e^{-A^{\mathcal{P}}(\tau-t)}X(t) \\
&+ 2\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right) + X(t)' e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3 Y_\tau + Y_\tau' \mathcal{M}_3 Y_\tau.
\end{aligned}$$

Now integrate the first term over the time horizon τ given $X(t) = X$.

$$\begin{aligned}
&-\frac{1}{2}\int_t^{t+\tau} X(s)'\mathcal{M}_3 X(s) ds = \\
&-\frac{1}{2}\int_t^{t+\tau} \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(s-t)}\right)\mathcal{M}_3\left(I - e^{-A^{\mathcal{P}}(s-t)}\right) ds \left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau}\left(I - e^{-A^{\mathcal{P}\prime}(s-t)}\right)\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)} ds X(t) - \frac{1}{2}X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)} ds X(t) \\
&-\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau}\left(I - e^{-A^{\mathcal{P}\prime}(s-t)}\right)\mathcal{M}_3 Y_s ds + X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 Y_s ds\right) - \frac{1}{2}\int_t^{t+\tau} Y_s' \mathcal{M}_3 Y_s
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau \\
&+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}ds\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}ds\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\int_t^{t+\tau} e^{-A^{\mathcal{P}}(s-t)}dsX(t)+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}dsX(t) \\
&-\frac{1}{2}X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}dsX(t)-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\int_t^{t+\tau} Y_s ds \\
&+\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 Y_s ds - X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 Y_s ds\right)-\frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s.
\end{aligned}
$$

If we use the definition of Y_s, we have

$$
\begin{aligned}
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\int_t^{t+\tau} Y_s ds+\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3\int_t^s e^{-A^{\mathcal{P}}(s-v)}\Sigma_X d\epsilon_v ds\right. \\
&\left.-X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3\int_t^s e^{-A^{\mathcal{P}}(s-v)}\Sigma_X d\epsilon_v ds\right)=0, \qquad (87)
\end{aligned}
$$

since $d\epsilon_v ds = 0$ by Ito's Rule.

We need the result

$$
\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}ds=\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right],
$$

where the matrix $\mathcal{M}$ solves the Lyapunov equation

$$
-A^{\mathcal{P}}\mathcal{M}-\mathcal{M}A^{\mathcal{P}\prime}=\mathcal{M}_3. \qquad (88)
$$

The solution to this equation is a positive definite symmetric matrix, which is easily calculated using lyap.m in Matlab.

$$
\begin{aligned}
&-\frac{1}{2}\int_t^{t+\tau} X(\tau)'\mathcal{M}_3 X(\tau)ds=-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau \\
&+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I-e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I-e^{-A^{\mathcal{P}}\tau}\right]X(t)+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t) \\
&-\frac{1}{2}X(t)'\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)-\frac{1}{2}\int_t^{t+\tau} Y_s\mathcal{M}_3 Y_s
\end{aligned}
$$

We also need

$$\begin{aligned}
\int_t^{t+\tau} \mathcal{M}_2 X_s ds =& \mathcal{M}_2 \int_t^{t+\tau} \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} ds - \mathcal{M}_2 \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-t)} ds \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} ds \\
& + \mathcal{M}_2 \int_t^{t+\tau} e^{-A^{\mathcal{P}}(s-t)} ds X(t) + \int_t^{t+\tau} \mathcal{M}_2 Y_s ds \\
=& \mathcal{M}_2 \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} \tau - \mathcal{M}_2 \left(A^{\mathcal{P}}\right)^{-1} \left[I - e^{-A^{\mathcal{P}}(\tau)}\right] \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} \\
& + \mathcal{M}_2 \left(A^{\mathcal{P}\prime}\right)^{-1} \left[I - e^{-A^{\mathcal{P}}(\tau)}\right] X(t) + \int_t^{t+\tau} \mathcal{M}_2 \int_t^{s} e^{-A^{\mathcal{P}}(s-v)} \Sigma_X d\epsilon_v ds \\
=& \mathcal{M}_2 \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} \tau - \mathcal{M}_2 \left(A^{\mathcal{P}}\right)^{-1} \left[I - e^{-A^{\mathcal{P}}(\tau)}\right] \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} \\
& + \mathcal{M}_2 \left(A^{\mathcal{P}}\right)^{-1} \left[I - e^{-A^{\mathcal{P}}(\tau)}\right] X(t).
\end{aligned}$$

The last step uses the rule $d\epsilon_v dt = 0$

We also need

$$\begin{aligned}
\int_t^{t+\tau} d\epsilon_s' \Sigma_X \mathcal{M}_5' X_s =& \int_t^{t+\tau} d\epsilon_s' \Sigma_X \mathcal{M}_5' \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} - \int_t^{t+\tau} d\epsilon_s' \Sigma_X \mathcal{M}_5' e^{-A^{\mathcal{P}}(\tau-t)} \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} \\
& + \int_t^{t+\tau} d\epsilon_s' \Sigma_X \mathcal{M}_5' e^{-A^{\mathcal{P}}(s-t)} X(t) + \int_t^{t+\tau} d\epsilon_s' \Sigma_X \mathcal{M}_5' Y_s \\
=& \int_t^{t+\tau} d\epsilon_s' \Sigma_X \left[\mathcal{M}_5' \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} - \mathcal{M}_5' e^{-A^{\mathcal{P}}(\tau-t)} \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}} \right. \\
& \left. + \mathcal{M}_5' e^{-A^{\mathcal{P}}(s-t)} X(t)\right] + \int_t^{t+\tau} d\epsilon_s' \Sigma_X \mathcal{M}_5' Y_s \\
=& \int_t^{t+\tau} \left(\mathbb{M}_4 + X_t' \mathbb{M}_5 + Y_s' \mathcal{M}_5 \Sigma_X'\right) d\epsilon_s
\end{aligned}$$

We now bring all these calculations into the stochastic process for the pricing kernel.

$$\frac{M_{\tau,t}}{M_{t,t}} = \exp\Bigg\{ -\mathcal{M}_1(\tau)\tau - \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau$$
$$+ \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I - e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$- \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$- \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right]X(t) + \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)$$
$$- \frac{1}{2}X(t)'\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t) + \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau - \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$+ \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]X(t) - \frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s + \int_t^{t+\tau}\left(\mathbb{M}_4 + X_t'\mathbb{M}_5 + Y_s'\mathcal{M}_5\Sigma_X'\right)d\epsilon_s\Bigg\}. \tag{89}$$

Define

$$\mathcal{M}(\tau, X) \equiv \exp\Bigg\{ -\mathcal{M}_1(\tau)\tau - \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau$$
$$+ \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I - e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$- \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$+ \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau - \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$+ \Bigg[\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right] + \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]$$
$$- \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right]\Bigg]X(t) - \frac{1}{2}X(t)'\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)\Bigg\}$$
$$= \exp\left\{ -\frac{1}{2}\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1\right\} \tag{90}$$

This result can be used to separate the portion of the pricing kernel dependent on the current factors X from future random changes in these factors Y_s for $s > t$. We substitute the known part (25) into the pricing kernel (24) so that

$$\frac{1}{\mathcal{M}(\tau, X)}\frac{M_{\tau,t}}{M_{t,t}} = \exp\left\{ -\frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s ds + \int_t^{t+\tau}\left(\mathbb{M}_4 + X_t'\mathbb{M}_5 + Y_s'\mathcal{M}_5\Sigma_X'\right)d\epsilon_s\right\} \tag{91}$$

This relation is an example of the stochastic process (44) so that its probability distribution is the solution to the forward Kolmogorov equation (13). Notice (91) is dependent on the current X through $\mathbb{M}_5$. This means that $\mathcal{D}_4 \equiv \mathbb{M}_4 + X_t'\mathbb{M}_5$ and $\mathcal{D}_5 = \mathcal{M}_5\Sigma_X'$. These terms do not influence the forward Kolmogorov equation, since this error term has mean zero.

The solution to the forward Kolmogorov equation yields the probability distribution for the pricing kernel.

$$\frac{1}{\mathcal{M}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}} \sim \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau,X))}} \exp\left\{ -\frac{1}{2}\mathcal{A}_1(\tau,X) - \frac{1}{2}Y'\mathcal{A}_3(\tau,X)^{-1}Y \right\}$$

which has the same form as (44) with the appropriate definitions of the coefficients $\mathcal{D}'s$.

Thus the probability distribution for the pricing kernel is given by

$$\frac{M_{\tau,t}}{M_{t,t}} \sim \exp\left\{ -\frac{1}{2}\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau) \right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau))}} \exp\left\{ -\frac{1}{2}Y'\mathcal{A}_3(\tau)^{-1}Y \right\}$$

This leads to equation (25) and (26) in the text with $\sigma_M \equiv \mathcal{A}_3(\tau)$.

$$E_t\left[\frac{M_{\tau,t}}{M_{t,t}}\right] = \exp\left\{ -\frac{1}{2}\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau) \right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau))}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}Y'\mathcal{A}_3(\tau)^{-1}Y \right\} dY$$
$$= \exp\left\{ -\frac{1}{2}\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau) \right\} \qquad (92)$$

This corresponds to equation (25) in the text with

$$\begin{aligned} \left(\sigma_{\mathcal{M}}(\tau)\right)^{-1} \equiv& \mathfrak{M}_3 \\ \mathcal{M}(\tau) \equiv& \exp\left\{ \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau) \right\}. \end{aligned} \qquad (93)$$

REFERENCES

Adrian, Tobias, Richard K. Crump, and Emanual Moench (2013) "Pricing the Term Structure with Linear Regressions," *Journal of Financial Economics* 110, 110-138.

Anderson, Leif, Darrell Duffie, and Yang Song (2018) "Funding Value Adjustments" forthcoming *Journal of Finance* and working paper Stanford Business School.

Ang, Andrew, and Monika Pizzesi (2003) "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics* 50,745-787.

Ang, Andrew, Geert Behaert, and Min Wei (2008) "The Term Structure of Real Rates and Expected Inflation," *Journal of Finance* 63, 797-849.

Ang, Andrew, Jean Boivin, Sen Dong, Rudy, and Loo-Kung (2008) "Monetary Policy Shifts and the Term Structure," *Review of Economic Studies* 78, 429-457.

Arnold Ludwig (1974) *Stochastic Differential Equations: Theory and Applications*, John Wiley and Sons.

Bauer, Michael D., and Glenn D. Rudebusch (2016) "Monetary Policy Expectations at the Zero Lower Bound," *Journal of Money, Credit and Banking* 48, 1439-1465.

Bauer, Michael D., and Glenn D. Rudebusch (2017) "Resolving the Spanning Puzzle in Macro-Finance Term Structure Models," *Review of Finance* 21, 511-553.

Calin, O., Chang, D-C, Furutani, K, and Iwasaki, C. (2011) *Heat Kernels for Elliptic and Sub-Elliptic Operators*, Birkhauser.

Chami, Ralph, Thomas F. Cosimano, Jun Ma, and Celine Rochon (2017), "What's Different about Bank Holding Companies?," *IMF Working Paper* 17/26.

Chami, Ralph, Thomas F. Cosimano, Celine Rochon, and Julieta Yung (2018), "Monetary Policy and the Term Structure," *Working Paper*.

Chernov, Mikhail, and Phillipe Mueller (2012) "The Term Structure of Expected Inflation," *Journal of Financial Economics* 106, 367-394.

Chirikjian, Gregory S., (2009) *Stochastic Models, Information Theory and Lie Groups*, Birkhäuser.

Cochrane, John, and Monica Piazzesi (2005) "Bond Risk Premia," *American Economic Review* 95, 138-160.

Cosimano, Thomas F. and Julieta Yung (2018) "Foreign Exchange and the Term Structure" working paper.

Dai, Qiang, and Kenneth Singleton (2000) "Specification Analysis of Affine Term Structure Models," *Journal of Finance* 55, 1943-1978.

Dai, Qiang, and Kenneth Singleton (2002) "Expectations Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure," *Journal of Financial Economics* 63, 415-441.

Duffie, Darrell (2001) *Dynamic Asset Pricing Theory*, Princeton University Press.

Duffie, Darrell, and Ming Huang (1996) "Swap Rates and Credit Quality," *Journal of Finance* 51, 921-949.

Duffie, Darrell, and Rui Kan (1996) "A Yield-Factor model of interest rates," *Mathematical Finance* 6, 379-406.

Duffie, Darrell, Jun Pan, and Kenneth Singleton (2000) "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica* 68,1343-1376.

Duffie, Darrell, and Kenneth Singleton (1997) "An Econometric Model of the Term Structure of Interest Rate Swap Yields," *Journal of Finance* 52, 1287-1323.

Durham, J. Benson (2013) "Arbitrage-Free Models of Stocks and Bonds" working paper Federal Reserve Bank of New York.

Durham, J. Benson (2015) "Arbitrage-Free Affine Models of the Forward Price of Foreign Currency" working paper Federal Reserve Bank of New York.

Greenwood, Robin, and Dimitri Vayanos (2014) "Bond Supply and Excess Bond Return," *Review of Financial Studies* 27, 663-713.

Grinblatt, Mark, and Francis LOngstaff (2000) "Financial Inovation and the Role of Derivative Securities: An Empirical Analysis of the Treasury STRIPS Program," *Journal of Finance* 55, 1415-1436.

Hamilton, James, and Jing Cynthia Wu (2012a) "Identification and Estimation of Gaussian Affine Term Structure Models," *Journal of Econometrics* 168, 315-331.

Hamilton, James, and Jing Cynthia Wu (2012b) "The Effectiveness of Alternative Monetary Policy Tools in a Zero Lower Bound Environment," *Journal of Money, Credit and Banking* 44, 3-46.

Hamilton, James, and Jing Cynthia Wu (2014a) "Testable Implications of Affine Term Structure Models," *Journal of Econometrics* 178, 231-242.

Hamilton, James, and Jing Cynthia Wu (2014b) "Risk Premia in Crude OIl Futures Prices" *Journal of International Money and Finance* 42, 9-37.

Hamilton, James, and Jing Cynthia Wu (2016) "Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bond," *Journal of Money, Credit and Banking* 48, 253-291.

Hijab, O. (1987) *Stabilization of Control Systems*, Springer-Verlag.

Joslin, Scott, Kenneth J. Singleton, and Haoxiang Zhu (2011) "A New Perspective on Gaussian Dynamic Term Structure Models," *Review of Financial Economics* 24, 2184-2227.

Joslin, Scott, Marcel Priebsch, and Kenneth J. Singleton (2014) "Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks," *Journal of Finance* 69, 1197-1233.

Karatzas, Ioannis, and Steven E. Shreve (1988) *Brownian Motion and Stochastic Calculus*, Springer-Verlag.

Kothari, S. P., Jonathan Lewellen, and Jerold B. Warner (2017) "The Behavior of Aggregate Corporate Investment working paper Sloan School of Management MIT.

Krippner, Leo (2015) *Zero Lower Bound Term Structure Modeling: A Practitioner's Guide*, Palgrare MacMillan.

Li, Canlin, and Min Wei (2013) "Term Structure Modeling with Supply Factors and the Federal Reserve's Large-Scale Asset Purchases Programs," *International Journal of Central Banking* 14, 3-40 .

Liu, Jun (2007) "Portfolio Selection in Stochastic Environments," *Review of Financial Studies* 20, 1-39

Liu, Linlin (2016) "Essays on Asset Returns and Portfolio Allocation," Chapter 2 of PhD Dissertation, University of Notre Dame.

Longstaff, Francis, Pedro Santa-Clara, and Edwardo Swartz (2001) "The Relative Valuation of Caps and Swaaptions Theory and Empirical Evidence," *Journal of Finance* 56, 2067-2109.

Øksendal Bernt (2005) *Stochastic Differential Equations: An Introduction with Applications* Springer-Verlag.

Petersen, Kaare B., and Michael S. Pedersen (November 14, 2008) "The Matrix Cookbook" at $http://matrixcookbook.com$.

Piazzesi, Monica (2010) "Affine Term Structure Models," *Handbook of Financial Econometrics* edited by Y. Aït-Sahalia L. H. Hansen 12, 691-766 , North Holland.

Protter, Philip E. (2005) *Stochastic Integration and Differential Equations*, second edition, Springer-Verlag.

Sangvinatsos, Antonios, and Jessica A. Wachter (2005) "Does the Failure of the Expectations Hypothesis Matter for Long-Term Investors?," *The Journal of Finance* 60, 179-230.

Shreve, Steven E. (2006) *Stochastic Calculus for Finance II: Continuous-Time Models* Springer-Verlag.

Strauss, Walter A. (2008) *Partial Differential Equations: An Introduction*, John Wiley and Sons.

Strichartz, Robert S. (2008), *A Guide to Distribution Theory and Fourier Transforms*, World Scientific.

Tevlin, Stacey, and Karl Whelan (2003) "Explaining the Investment Boom of the 1990s," *Journal of Money, Credit and Banking* 35, 1-22.

Wu, Jing Cynthia, and Fan Dora Xia (2016) "Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound," *Journal of Money, Credit and Banking* 48, 253-291.

Yung, Julieta (2017) Can Interest Rate Factors Explain Exchange Rate Fluctuations? working paper Bates College.